

Vizing's conjecture for cographs

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ABSTRACT. We show that if G is a cograph, that is P_4 -free, then for any graph H , $\gamma(G \square H) \geq \gamma(G)\gamma(H)$. By the characterization of cographs as a finite sequence of unions and joins of K_1 , this result easily follows from that of Bartsalkin and German. However, the techniques used are new and may be useful to prove other results.

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1. Introduction

Vizing's conjecture [12], now open for fifty-three years, states that for any two graphs G and H ,

$$(1.1) \quad \gamma(G \square H) \geq \gamma(G)\gamma(H)$$

where $\gamma(G)$ is the domination number of G .

The survey [4] discusses many results and approaches to the problem. For more recent partial results see [11], [10], [3], [6], [8], and [9].

A predominant approach to the conjecture has been to show it true for some large class of graphs. For example, in their seminal result, Bartsalkin and German [2] showed the conjecture for decomposable graphs. More recently, Aharoni and Szabó [1] showed the conjecture for chordal graphs and Brešar [3] gave a new proof of the conjecture for graphs G with domination number 3.

We say that a bound is of *Vizing-type* if $\gamma(G \square H) \geq c\gamma(G)\gamma(H)$ for some constant c , which may depend on G or H . It is known [11] that all graphs satisfy the Vizing-type bound,

$$\gamma(G \square H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}.$$

Restricting the graphs, but as a generalization of Bartsalkin and German's class of decomposable graphs, Contractor and Krop [6] showed

$$\gamma(G \square H) \geq \left(\gamma(G) - \sqrt{\gamma(G)}\right)\gamma(H)$$

where G belongs to \mathcal{A}_1 , the class of graphs which are spanning subgraphs of domination critical graphs G' , so that G and G' have the same domination number and the clique partition number of G' is one more than its domination number.

Krop [8] showed that any claw-free graph G satisfies the Vizing-type bound

$$\gamma(G \square H) \geq \frac{2}{3} \gamma(G) \gamma(H)$$

In this paper we show that the class of induced P_4 -free graphs, or cographs, satisfies Vizing's conjecture.

1.1. Notation. All graphs $G(V, E)$ are finite, simple, connected, undirected graphs with vertex set V and edge set E . We may refer to the vertex set and edge set of G as $V(G)$ and $E(G)$, respectively. For more on basic graph theoretic notation and definitions we refer to Diestel [7].

For any graph $G = (V, E)$, a subset $S \subseteq V$ *dominates* G if $N[S] = G$. The minimum cardinality of $S \subseteq V$, so that S dominates G is called the *domination number* of G and is denoted $\gamma(G)$. We call a dominating set that realizes the domination number a γ -set.

The *Cartesian product* of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, denoted by $G_1 \square G_2$, is a graph with vertex set $V_1 \times V_2$ and edge set $E(G_1 \square G_2) = \{((u_1, v_1), (u_2, v_2)) : v_1 = v_2 \text{ and } (u_1, u_2) \in E_1, \text{ or } u_1 = u_2 \text{ and } (v_1, v_2) \in E_2\}$.

A graph G is a *cograph* or P_4 -free if it contains no induced P_4 subgraph.

Let G be any graph and S a subset of its vertices. Chellali et al. [5] defined S to be a $[j, k]$ -set if for every vertex $v \in V - S$, $j \leq |N(v) \cap S| \leq k$. Clearly, a $[j, k]$ -set is a dominating set. For $k \geq 1$, the $[1, k]$ -domination number of G , written $\gamma_{[1, k]}(G)$, is the minimum cardinality of a $[1, k]$ -set in G . A $[1, k]$ -set with cardinality $\gamma_{[1, k]}(G)$ is called a $\gamma_{[1, k]}(G)$ -set.

If $\Gamma = \{v_1, \dots, v_k\}$ is a minimum dominating set of G , then for any $i \in [k]$, define the set of *private neighbors* for v_i , $P_i = \{v \in V(G) - \Gamma : N(v) \cap \Gamma = \{v_i\}\}$. For $S \subseteq [k]$, $|S| \geq 2$, we define the *shared neighbors* of $\{v_i : i \in S\}$, $P_S = \{v \in V(G) - \Gamma : N(v) \cap \Gamma = \{v_i : i \in S\}\}$.

For any $S \subseteq [k]$, say $S = \{i_1, \dots, i_s\}$ where $s \geq 2$. We may write P_S as $P_{\{i_1, \dots, i_s\}}$ or P_{i_1, \dots, i_s} interchangeably.

For $i \in [k]$, let $Q_i = \{v_i\} \cup P_i$. We call $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ the *cells* of G . For any $I \subseteq [k]$, we write $Q_I = \bigcup_{i \in I} Q_i$ and call $\mathcal{C}(\bigcup_{i \in I} Q_i) = \bigcup_{i \in I} Q_i \cup \bigcup_{S \subseteq I} P_S$ the *chamber* of Q_I . We may write this as \mathcal{C}_I .

For a vertex $h \in V(H)$, the G -fiber, G^h , is the subgraph of $G \square H$ induced by $\{(g, h) : g \in V(G)\}$.

For a minimum dominating set D of $G \square H$, we define $D^h = D \cap G^h$. Likewise, for any set $S \subseteq [k]$, $P_S^h = P_S \times \{h\}$, and for $i \in [k]$, $Q_i^h = Q_i \times \{h\}$. By v_i^h we mean the vertex (v_i, h) . For any $I^h \subseteq [k]$, where I^h represents the indices of some cells in G -fiber G^h , we write \mathcal{C}_{I^h} to mean the chamber of $Q_{I^h}^h$, that is, the set $\bigcup_{i \in I^h} Q_i \cup \bigcup_{S \subseteq I^h} P_S^h$.

We may write $\{v_i : i \in I^h\}$ for $\{v_i^h : i \in I^h\}$ when it is clear from context that we are talking about vertices of $G \square H$ and not vertices of G .

For clarity, assume that our representation of $G \square H$ is with G on the x -axis and H on the y -axis.

Any vertex $v \in V(G) \times V(H)$ is *vertically dominated* if $(\{v\} \times N_H[h]) \cap D \neq \emptyset$ and *vertically undominated*, otherwise. For $i \in [k]$ and $h \in V(H)$, we say that the cell Q_i^h is *vertically dominated* if $(Q_i \times N_H[h]) \cap D \neq \emptyset$. A cell which is not vertically dominated is *vertically undominated*.

In our argument, we label vertices of a minimum dominating set D of $G \square H$, by labels from $[k]$ so that for any $i \in [k]$, projecting the vertices labeled by i onto H produces a dominating set of H . We call a vertex $(x, h) \in D^h$ with the single label i , *free*, if there exists another vertex $(y, h) \in D^h$, which is given the label i .

2. Cographs

THEOREM 2.1. *For any cograph G and any graph H , $\gamma(G \square H) \geq \gamma(G)\gamma(H)$.*

PROOF. Let $\Gamma = \{v_1, \dots, v_k\}$ be a minimum $[1, 2]$ dominating set of G and let D be a minimum dominating set of $G \square H$. By the result of Chellali et al. [5] (Theorem 8), $\gamma(G) = k$. Suppose $u \in V(G) - \{\Gamma\}$ is adjacent to two vertices of Γ , say v_1 and v_2 . Notice that if neither v_1 nor v_2 have private neighbors with respect to Γ , then we could replace v_1 and v_2 by u in Γ and produce a smaller dominating set, which is a contradiction. Hence, at least one of P_1 or P_2 is nonempty.

CLAIM 2.2. *There exists a vertex in $P_1 \cup P_2$ which is independent from both u and $V(G) - \{Q_1 \cup Q_2\}$.*

PROOF. Case 1: Suppose $P_1 \neq \emptyset$ and $P_2 = \emptyset$. Note that by the minimality of Γ no vertex of $\Gamma - \{v_2\}$ can be adjacent to v_2 . If $w_1 \in P_1$, then by definition of private neighbors, no vertex of $\Gamma - \{v_1\}$ is adjacent to w_1 . If u is not adjacent to w_1 , then we produce $P_4 : w_1 v_1 u v_2$ which contradicts the definition of G . However, if u is adjacent to every vertex of P_1 , then we could replace v_1 and v_2 by u in Γ which would produce a smaller dominating set, which is impossible.

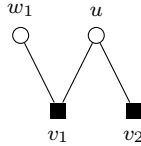


FIGURE 1.

Case 2: Suppose $P_1, P_2 \neq \emptyset$. By minimality of Γ , some vertex of $P_1 \cup P_2$ is not adjacent to u . Suppose such a vertex is $w_2 \in P_2$. We may assume v_1 is adjacent to v_2 , else we would produce $P_4 : w_2 v_2 u v_1$. For any vertex $w_1 \in P_1$, we may assume w_1 is adjacent to w_2 , else we would produce $P_4 : w_1 v_1 v_2 w_2$. Notice u is adjacent to w_1 to avoid $P_4 : w_2 w_1 v_1 u$. Suppose $u' \in V(G) - \{Q_1 \cup Q_2\}$ is adjacent to w_2 and suppose without loss of generality that u' is adjacent to v_3 .

Thus, we are left with the situation illustrated in Figure 2 where the drawn edges have been shown to exist.

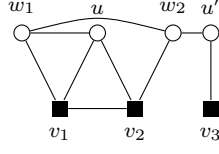


FIGURE 2.

Since u' may adjacent to at most two vertices of Γ we argue that u' is adjacent to either v_1 or v_2 , since otherwise we have $P_4 : u'w_2v_2v_1$.

Subcase (i): If u' is adjacent to v_2 , then u' is also adjacent to w_1 to avoid $P_4 : v_3u'w_2w_1$. Furthermore, if v_3 is not adjacent to v_1 or v_1 , then we produce $P_4 : v_3u'v_2v_1$. If v_3 is adjacent to v_1 , then we produce $P_4 : w_2v_2v_1v_3$. Thus, v_2 is adjacent to v_3 . However, now we have $P_4 : v_3v_2uw_1$ which is impossible.

Subcase (ii): If u' is adjacent to v_1 , then u' is also adjacent to w_1 to avoid $P_4 : v_3u'w_2w_1$. Furthermore, v_3 is adjacent to v_2 , else we have $P_4 : v_3u'w_2v_2$. However, this forces $P_4 : v_3v_2uw_1$ which is impossible. \square

For any $h \in V(H)$, suppose the fiber G^h contains $\ell_h (= \ell)$ vertically undominated cells $U^h = \{Q_{i_1}^h, \dots, Q_{i_\ell}^h\}$ for $0 \leq \ell \leq k$. We set $I^h = \{i_1, \dots, i_\ell\}$. Notice that for $j_1, j_2 \in [k] - I^h$, no vertex of P_{j_1, j_2}^h may dominate any of $v_{i_1}, \dots, v_{i_\ell}$. Thus, $\{v_i : i \in I^h\}$ must be dominated horizontally in G^h either by shared neighbors of $\{v_i : i \in I^h\}$ or by vertices of $\{v_i : 1 \leq i \leq k, i \notin I^h\}$. Furthermore, the private neighbors $\{P_i^h : i \in I^h\}$ must be dominated horizontally in G^h either by shared neighbors of $\{v_i : i \in I^h\}$ or by vertices of $\{P_i^h : 1 \leq i \leq k, i \notin I^h\}$.

We label the vertices of D by the following *Provisional Labeling*. If a vertex of D^h for any $h \in H$, is in Q_i^h for $1 \leq i \leq k$, then we label that vertex by i . If v is a shared neighbor of some subset of $\{v_i : i \in I^h\}$, then it is a member of $P_{i,j}^h$ for some $i, j \in I^h$, and we label v by the pair of labels (i, j) . If v is a member of $P_{i,j}^h$ for $i \in I^h$ and $j \in [k] - I^h$, then we label v by i . If v is a member of $P_{i,j}^h$ for $i, j \in [k] - I^h$, then we label v by either i or j arbitrarily.

After the labels are placed, all vertices of D have a single label or a pair of labels.

Next, we apply a relabeling to some of the vertices of D which we call the *First Refinement*. For a fixed $h \in H$, suppose v is some shared neighbor of two vertices of $\{v_i : i \in I^h\}$ in the chamber of $Q_{I^h}^h$, which is vertically dominated, say by $y \in D^{h'}$ for some $h' \in H$, $h \neq h'$. In other words, we suppose $v \in P_{j_1, j_2}^h$ for some $j_1, j_2 \in I^h$ which implies that $y \in P_{j_1, j_2}^{h'}$.

The vertex y may be labeled by one or two labels, regardless of whether the First Refinement had been performed on $D^{h'}$.

Suppose that y is labeled by one label, say j_1 . If D^h contains a vertex $x \in P_{j_1, j_2}^h$, then we remove the pair of labels (j_1, j_2) from x and relabel x by j_2 .

Suppose y is labeled by the pair of labels, (j_1, j_2) . If D^h contains a vertex $x \in P_{j_1, j_2}^h$, then we remove the pair of labels (j_1, j_2) from x and then relabel x arbitrarily by one of the single labels j_1 or j_2 .

After the labeling, a vertex v of D may have a pair of labels (i, j) if $v \in P_{i, j}^h$ and for any $h' \in N_H(h)$, $D^{h'} \cap P_{i, j}^{h'} = \emptyset$.

Finally, we relabel some of the vertices of D by the *Second Refinement*. For every $h \in H$, if D^h contains vertices x and y with pairs of labels $(j_1, j_2), (j_2, j_3)$ respectively, for some integers j_1, j_2 , and j_3 , then we relabel y by the label j_3 . If x and y are labeled j_1 and (j_1, j_2) respectively, for some integers j_1, j_2 , we relabel y by j_2 . We apply this relabeling to pairs of vertices of D^h , sequentially, in any order.

CLAIM 2.3. *After the Second Refinement every label on a vertex of D is a single label.*

PROOF. For any $h \in V(H)$, suppose $v \in D^h$ has a pair of labels (i, j) . The Provisional Labeling prescribes that $i, j \in I^h$ which means that Q_i and Q_j are vertically undominated cells. If there exists $w \in D^h \cap P_{j, m}$ for any $1 \leq m \leq k$, or $x \in D^h \cap P_j$, then v would have a single label after the Second Refinement which is not the case. By Claim 2.2, some vertex x in $P_i^h \cup P_j^h$ is independent from v and independent from $V(G) - \{Q_i \cup Q_j\}$. However, this means that x is undominated, which contradicts the fact that D is a dominating set. \square

Suppose that for some $h \in V(H)$, G^h contains a cell, Q_i^h , which is vertically undominated and the vertices of D^h dominating Q_i^h are not labeled i . In this case, v_i^h can only be dominated by other members of $\{v_j^h : j \in [k], j \neq i\}$, so suppose for some $j_1 \neq i, j_1 \in [k]$, there exists $v_{j_1}^h \in D^h$ so that v_i is adjacent to v_{j_1} . To avoid a contradiction to the minimality of Γ , we see that $P_i \neq \emptyset$ and say $u \in P_i^h$. Notice that if u is dominated by some $u' \in P_{j_2} \cap D^h$ for some $j_2 \neq i, j_1$, then we produce $P_4 : v_{j_1} v_i u u'$ in G^h and thus in G . Furthermore, if $v \in P_{j_1} \cap D^h$ dominates u , then v is a free vertex labeled j_1 and we may relabel v by i without changing the vertically dominated status of cells $Q_{j_1}^{h'}$ for any $h' \in V(H)$. Finally, suppose u is dominated by some shared neighbor $w \in P_{j_1, j_2} \cap D^h$. Notice if $x \in P_{j_1}$, then we produce $P_4 : x v_{j_1} v_i v_{j_2}$ and if $y \in P_{j_2}$, then we produce $P_4 : y v_{j_2} v_i v_{j_1}$ which cannot occur. Thus, $P_{j_1} = P_{j_2} = \emptyset$. If for every $w' \in P_{j_1, j_2}$, w is adjacent to w' , we have a contradiction to the minimality of Γ , since now we can replace v_{j_1} and v_{j_2} by the projection of w onto $V(G)$ and form a smaller dominating set of G . We are left to suppose there exists $w' \in P_{j_1, j_2}$ so that w' is not adjacent to w . To avoid $P_4 : w' v_{j_1} w u$, w' must be adjacent to u . Furthermore, this same property is true for any vertex $v \in P_{j_1, j_2}$ not adjacent to w' or w , namely, v must be adjacent to u . To avoid $P_4 : v_{j_2} w' v_{j_1} v_i$ we must also have v_{j_2} adjacent to v_i . At this point, notice that $\{v_{j_1}, v_{j_2}, P_{j_1, j_2}\}$ is dominated by v_i and u , which is a contradiction to the

minimality of Γ , since now we can replace v_{j_1} and v_{j_2} in Γ by the projection of u onto $V(G)$ and produce a smaller dominating set of G .

Notice that for any $i \in [k]$, projecting all vertices with a label i to H produces a dominating set of H . Summing over all i , we count at least $\gamma(G)\gamma(H)$ vertices in D .

□

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